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1980 J. Phys. A: Math. Gen. 13 933

(http://iopscience.iop.org/0305-4470/13/3/024)

J. Phys. A: Math. Gen. 13 (1980) 933–938. Printed in Great Britain

Electromagnetic fields in the Gödel universe

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Received 31 January 1979, in final form 12 September 1979

Abstract. Perturbative electromagnetic fields are investigated in the Gödel universe using the Debye potential (two-component Hertz potential) formalism. The Gödel space–time lends itself to this approach since it is algebraically special (Petrov D). With the usual system of coordinates used in the text describing the Gödel universe, the three of the basis coordinate vectors $\partial/\partial x^0$, $\partial/\partial x^1$ and $\partial/\partial x^3$ which are Killing, the field at infinity in these directions remains oscillatory, while for the remaining non-Killing coordinate x^2 the field decays at both extremeties. This result compares dramatically well with the behaviour of the null geodesics in the Gödel universe.

1. Introduction

The computation of perturbative electromagnetic fields in curved space–times is of considerable interest in general relativity. Cohen and Kegeles (1974, referred to hereafter as I) have given a prescription for decoupling the Maxwell equations in algebraically special space–times by utilising the Hertz potential formalism. The prescription furnishes a decoupled equation for the Debye potentials (two-component Hertz potential). The equation is derived using the Newman–Penrose formalism. The actual electromagnetic fields (except the monopole field $l = 0$) may be obtained by some simple operations of differentiations of the Debye potentials. Cohen and Kegeles have used this approach successfully for the Schwarzschild, Kerr and Robertson–Walker space–times.

In this paper we study the behaviour of electromagnetic fields in the Gödel universe. The Gödel solution lends itself to this type of investigation as it is algebraically special—type [2, 2]—and devoid of any intrinsic electromagnetic field (it is not a solution of the Einstein–Maxwell equations). In § 2 we write down the null tetrad for the Gödel metric and then list the spin coefficients for the tetrad. The scheme for this computation is given in detail in I. We state the equation for the Debye potentials also given there. In § 3 the equation is solved exactly in terms of known functions and the results compared with the classical approach involving null geodesics.

2. The governing equation for the Debye potentials

The Gödel universe is a solution of Einstein's equations (with non-zero cosmological

‡ Supported by an NSF grant.

constant) with the source as dust. Its geometry is described by the line element:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 - \tfrac{1}{2} e^{2qx^2}(dx^3)^2 - 2 e^{qx^2} dx^0 dx^3 \tag{2.1}$$

where q is a parameter related to the vorticity of the fluid.

We adopt the Newman–Penrose formalism for our investigations. The notation is identical to that of I. We start by writing down the null tetrad for the above line element given in (2.1):

$$\begin{aligned} k_a &= \frac{1}{\sqrt{2}}(1, -1, 0, e^{qx^2}) \qquad & n_a &= \frac{1}{\sqrt{2}}(1, 1, 0, e^{qx^2}) \\ m_a &= \frac{1}{\sqrt{2}}\left(0, 0, 1, \frac{i}{\sqrt{2}} e^{qx^2}\right) \qquad & \bar{m}_a &= \frac{1}{\sqrt{2}}\left(0, 0, 1, -\frac{i}{\sqrt{2}} e^{qx^2}\right). \end{aligned} \tag{2.2}$$

The tetrad vectors satisfy the following relations:

$$k^a n_a = -m_a \bar{m}^a = -\bar{m}_a m^a = -1. \tag{2.3}$$

All other scalar products vanish. It is seen that k^a and n^a are repeated principal null directions of the Weyl tensor and hence from the Göldberg–Sachs theorem their congruences are geodesic and shear-free. This may also be seen immediately by an explicit calculation of the spin coefficients of the null tetrad. Therefore we now proceed to evaluate the spin coefficients and then write down the equation for the Debye potentials.

We use the obvious correspondence between the numerical indexing and the contravariant tetrad vectors,

$$k^a, n^a, m^a, \bar{m}^a \leftrightarrow 1, 2, 3, 4.$$

The intrinsic frame derivatives and the tangent vectors may be simultaneously given in terms of k^a, n^a, m^a and $\bar{m}^a$.

$$\begin{aligned} w_1 &= D = k^a \partial/\partial x^a \qquad & w_2 &= \Delta = n^a \partial/\partial x^a \\ w_3 &= \delta = m^a \partial/\partial x^a \qquad & w_4 &= \bar{\delta} = \bar{m}^a \partial/\partial x^a. \end{aligned} \tag{2.4}$$

The dual 1-forms defined by $w^i(w_j) = \delta^i_j$ may be obtained from the scalar product relations (2.3):

$$w^1 = -n_a\, dx^a \qquad w^2 = -k_a\, dx^a \qquad w^3 = \bar{m}_a\, dx^a \qquad w^4 = m_a\, dx^a. \tag{2.5}$$

The computation of the spin coefficients is done as in I. Here, we merely state the results of such a calculation:

$$\begin{aligned} \alpha &= -\frac{1}{2\sqrt{2}} q \qquad & \beta &= \frac{1}{2\sqrt{2}} q \qquad & \gamma &= -\frac{i}{4} q \\ \epsilon &= -\frac{i}{4} q \qquad & \mu &= -\frac{i}{2} q \qquad & \rho &= -\frac{i}{2} q. \end{aligned} \tag{2.6}$$

All other spin coefficients vanish.

The intrinsic frame derivatives may be stated in terms of the standard basis vectors $\partial/\partial x^i$, $i=0, 1, 2, 3$:

$$D=\frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x^0}-\frac{\partial}{\partial x^1}\right) \qquad \Delta=\frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^1}\right)$$
$$\delta=-\mathrm{i}\frac{\partial}{\partial x^0}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial x^2}+\mathrm{i}\,\mathrm{e}^{-qx^2}\frac{\partial}{\partial x^3} \tag{2.7}$$
$$\bar{\delta}=\mathrm{i}\frac{\partial}{\partial x^0}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial x^2}-\mathrm{i}\,\mathrm{e}^{-qx^2}\frac{\partial}{\partial x^3}.$$

The decoupled equation for the complex scalar potential ψ, as given in I, is stated in terms of the intrinsic frame derivatives and the spin coefficients:

$$[-(\Delta-\gamma-\bar{\gamma}+\bar{\mu}-\mu)D+(\bar{\delta}-\alpha+\bar{\beta}-\pi-\bar{\tau})\delta]\psi=0. \tag{2.8}$$

Substitution of the spin coefficients from (2.7) and the intrinsic frame derivatives (2.8) furnishes a decoupled equation for ψ in the case of the Gödel universe:

$$\left(\frac{\partial^2}{\partial(x^0)^2}+\frac{\partial^2}{\partial(x^1)^2}+\mathrm{i}\sqrt{2}q\frac{\partial}{\partial x^1}+\frac{\partial^2}{\partial(x^2)^2}+q\frac{\partial}{\partial x^2}\right.$$
$$\left.-4\,\mathrm{e}^{-qx^2}\frac{\partial^2}{\partial x^0\,\partial x^3}+2\,\mathrm{e}^{-qx^2}\frac{\partial^2}{\partial(x^3)^2}\right)\psi=0. \tag{2.9}$$

The tetrad components of the Maxwell field tensor in terms of ψ are the following:

$$\phi_0=f_{km}=[(\delta-\bar{\alpha}-\beta-\bar{\pi})D+(D-\epsilon-\bar{\epsilon}-\bar{\rho})\delta]\bar{\psi}$$
$$\phi_1=\tfrac{1}{2}(f_{kn}+f_{\bar{m}m})=[(\Delta-\gamma-\bar{\gamma}+\bar{\mu}-\mu)D+(\bar{\delta}-\bar{\alpha}+\beta+\bar{\pi}+\tau)\bar{\delta}]\bar{\psi} \tag{2.10}$$
$$\phi_2=f_{\bar{m}n}=[(\Delta+\gamma-\bar{\gamma}+\bar{\mu})\bar{\delta}+(\bar{\delta}+\alpha+\bar{\beta}-\bar{\tau})\Delta]\bar{\psi}.$$

In the standard basis of the coordinate vectors the tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu}=2(\phi_1+\bar{\phi}_1)n_{[\mu}k_{\nu]}+2\phi_2k_{[\mu}m_{\nu]}+2\bar{\phi}_2k_{[\mu}\bar{m}_{\nu]}$$
$$+2\phi_0\bar{m}_{[\mu}n_{\nu]}+2\bar{\phi}_0m_{[\mu}n_{\nu]}+2(\phi_1-\bar{\phi}_1)m_{[\mu}\bar{m}_{\nu]}. \tag{2.11}$$

The solution of equation (2.9) for ψ is sufficient to determine the entire electromagnetic field (except for the monopole field $l=0$) from the relations (2.10) and (2.11). The next section we devote to the solution of the equation and try to extract as much information from it as possible.

3. The solution of the Debye potential equation

Due to inherent symmetries present in the geometry of the Gödel universe, the problem of solution is simplified. The Gödel universe possesses five Killing vectors of which the three obvious ones $\partial/\partial x^0$, $\partial/\partial x^1$ and $\partial/\partial x^3$ provide us with a simplification of the equation. The equation (2.9) may be solved by the separation of variables, assuming ψ to be of the form

$$\psi=\exp(\pm\mathrm{i}k_0x^0\pm\mathrm{i}k_1x^1\pm\mathrm{i}k_3x^3)Z(x^2) \tag{3.1}$$

where k_0, k_1 and k_3 are positive constants and $Z(x^2)$ is to be determined from the equation obtained by substituting the form of the solution (3.1) into the equation (2.9). The equation for $Z(x^2)$ is

$$\frac{d^2Z}{d(x^2)^2}+q\frac{dZ}{dx^2}+(-2k_3^2\,e^{-2qx^2}+4k_0k_3\,e^{-qx^2}-k_0^2-k_1^2-\sqrt{2}qk_1)Z=0. \tag{3.2}$$

The equation (3.2) appears a little cumbersome in the coordinate x^2. A transformation to a more convenient variable $u=e^{-qx^2}$ gives the equation for Z as

$$\frac{d^2Z}{du^2}+\left(-2\bar{k}_3^2+\frac{4\bar{k}_0\bar{k}_3}{u}-\frac{\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1}{u^2}\right)Z=0 \tag{3.3}$$

where $\bar{k}_i=k_i/q$, $i=0$, 1 and 3.

This is Kummer's equation and can be converted to the confluent hypergeometric equation. But, before doing so, one can obtain some asymptotic information from equation (3.3) itself. For large values of u, i.e. large negative values of x^2, the last two terms in the parenthesis may be neglected to give

$$d^2Z/du^2-2\bar{k}_3^2Z=0. \tag{3.4}$$

The solution of this equation is of the form $Z=\exp(\pm\sqrt{2}\bar{k}_3u)$. We disregard the solution with the positive sign as unphysical, for as $u\to\infty$, $Z(u)$ grows arbitrarily large, and as we are considering perturbative electromagnetic fields, this would be contrary to our assumption. One then chooses the solution with the negative sign. The solution decays exponentially at large negative values of x^2 which correspond to large positive values of u. The potential is unrestricted in the other three coordinates, the anomalous behaviour occurring only with the variation of the x^2 coordinate. In the other direction of $x^2\to\infty$, information may be sought from equation (3.3). Since $u\to0$, the dominant factor in the term multiplying $Z(u)$ is $-(\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1)/u^2$ which is negative and also causes a decay of the potential function. The solution is oscillatory in behaviour only when

$$-2\bar{k}_3^2+\frac{4\bar{k}_0\bar{k}_3}{u}-\frac{\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1}{u^2}>0 \tag{3.5}$$

and this would occur for intermediate values of u in the range of $(0,\infty)$ if $\bar{k}_0$ were sufficiently large. More precisely, if $\bar{k}_0^2>\bar{k}_1^2+\sqrt{2}\bar{k}_1$ the solution would be oscillatory in the region defined by

$$\bar{k}_0-[\tfrac{1}{2}(\bar{k}_0^2-\bar{k}_1^2-\sqrt{2}\bar{k}_1)]^{1/2}<\bar{k}_3u<\bar{k}_0+[\tfrac{1}{2}(\bar{k}_0^2-\bar{k}_1^2-\sqrt{2}\bar{k}_1)]^{1/2}. \tag{3.6}$$

This corresponds to the 'central' region of the x^2 coordinate. A large value of $\bar{k}_3$ makes the region narrower as may be seen from equation (3.6). Hence the field seems to be bunched up around the region $x_2\simeq0$ and dies out at both extremeties. For the other coordinates there is no such decay at infinity, and the solution always remains oscillatory. However, if we choose the negative sign for k_3 the oscillatory behaviour of the solution is absent.

The equation (3.3) converted to the confluent hypergeometric equation affords more accurate information regarding the field.

Setting $Z(u)=u^{n+1}\,e^{-2\bar{k}_3u}f(u)$ with n defined by the relation

$$n(n+1)=\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1 \tag{3.7}$$

and defining a new variable $v=2\sqrt{2}\bar{k}_3u$, f satisfies the confluent hypergeometric equation:

$$v\frac{d^2f}{dv^2}+[2(n+1)-v]\frac{df}{dv}-(n+1-\sqrt{2}\bar{k}_0)f=0. \tag{3.8}$$

The two independent solutions of (3.8) are ${}_1F_1(n+1-\sqrt{2}\bar{k}_0, 2n+2, v)$ and $v^{-2n-1}\,{}_1F_1(-n-\sqrt{2}\bar{k}_0, -2n, v)$. We use a linear combination of these solutions which satisfies the usual boundary condition that the solution $Z(v)$ decays when $x^2\to\pm\infty$. The linear combination which satisfies these conditions is

$$\begin{aligned}U(n&+1-\sqrt{2}\bar{k}_0, 2n+2, v)\\&=\frac{\Gamma(-2n-1)}{\Gamma(-n-\sqrt{2}\bar{k}_0)}\,{}_1F_1(n+1-\sqrt{2}\bar{k}_0, 2n+2, v)\\&\quad+\frac{\Gamma(2n+1)}{\Gamma(n+1-\sqrt{2}\bar{k}_0)}v^{-2n-1}\,{}_1F_1(-n-\sqrt{2}\bar{k}_0, -2n, v)\end{aligned} \tag{3.9}$$

with n the negative root of the equation (3.7). This may at once be seen from the following asymptotic behaviour:

$$Z(v)=v^{n+1}\,e^{-v/2}(v)$$

when $x^2\to\infty$ or, that is when $v\to 0$, $Z(v)\sim v^{-n}\,e^{-v/2}$ when $x^2\to-\infty$ or, that is when $v\to\infty$,

$$Z(v)\sim e^{-v/2}v^{\sqrt{2}\bar{k}_0}.$$

The latter asymptotic form of U may be also used in the event of $\bar{k}_3\gg 1$ when $x^2\simeq 0$. For in this case also $v\gg 1$ and the above approximation is valid. This would afford information about the field in the central region ($x^2\simeq 0$). The potential function then in this case has appreciable values predominantly in the region $x^2>0$. For $x^2<0$ the field dies out rapidly. For large values of $\bar{k}_0$ or $\bar{k}_1$, i.e. when the expression $\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1$ is large, $n\sim-(\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1)^{1/2}$ and the solution would die out as v^λ as $v\to 0$ or $x^2\to\infty$, where $\lambda=(\bar{k}_0^2+\bar{k}_1^2+\sqrt{2}\bar{k}_1)^{1/2}$.

It is interesting to compare the results obtained above with the null geodesics of the Gödel universe, since the electromagnetic radiation propagates along null geodesics in the high-frequency limit. Hence it would seem worthwhile to consider the above equations in this limit. The computation of the null geodesics is once again assisted by the existence of the three Killing vectors $\partial/\partial x^0$, $\partial/\partial x^1$ and $\partial/\partial x^3$. If u^i is a tangent vector to the geodesic then the corresponding covariant components have a constant value along a fixed null geodesic. Therefore we set

$$u_0=-p_0 \qquad u_1=p_1 \qquad u_3=-p_3.$$

The contravariant components of the tangent vector may be obtained by the metric tensor and substituted into the line element to get an equation for the remaining component u^2.

$$(u^2)^2=-p_0^2-p_1^2+4p_0p_3\,e^{-qx^2}-2\,e^{-2qx^2}p_3^2. \tag{3.10}$$

One notes that for large values of $|x^2|$ the expression on the right-hand side of (3.10) is negative, which means that real solutions do not exist in these regions. The null geodesics seem confined to a bounded range of the x^2 coordinate. The result compares favourably with the wave solution. A real solution for the null geodesic is only possible if the condition

$$4p_0p_3\,\mathrm{e}^{-qx^2}-p_0^2-p_1^2-2\,\mathrm{e}^{-2qx^2}p_3^2>0 \tag{3.11}$$

is satisfied. There seems to be a striking similarity between the expressions (3.5) and (3.11) if each $\bar{k}_i$ in (3.5) is replaced by p_i, $i=0$, 1 and 3. The difference occurs due to the term $\sqrt{2}\bar{k}_1$ in the expression of (3.5) which is absent in (3.11). This could be attributed to the fact that the former approach is the more accurate result and the latter is just its geometric optic approximation. Indeed, if $\bar{k}_0$ is large, i.e. in the high-frequency limit, the results become identical.

4. Conclusion

The Debye potential formalism is used to compute the electromagnetic fields in the Gödel universe. It is seen that the Debye potential equation can be solved by separation of variables because of the high symmetry present in the Gödel universe. The oscillations or decay occur only along the x^2 coordinate; in the remaining three independent directions the solution is always oscillatory. For sufficiently large values of $|x^2|$, there is a decay in the field strength so that the field appears to be predominant only in the central region of the x^2 coordinate. This fact compares favourably with the null geodesics which are also confined to almost the identical region of space-time. The regions under discussion coincide in the high-frequency limit ($\bar{k}_0 \gg 1$). Physically the difference in both the results occurs due to the low-frequency electromagnetic waves departing from the geodetic path, a phenomenon possibly similar to diffraction.

The Debye potential method that has been used in this paper can be extended to space-times with local rotational symmetry, the Gödel universe being a specific example of this class. These calculations will be published elsewhere (Dhurandhar *et al* 1979).

Acknowledgments

One of us (CVV) wishes to thank Professors Peter Havas and Jeffrey M Cohen for hospitality at Temple University and the University of Pennsylvania, respectively, during the summer of 1978. This visit was made possible by a NSF grant.

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